

5.1 Let \mathcal{M} be a differentiable manifold and ∇ a connection on \mathcal{M} .

- (a) Show that there exists no (1, 2)-type tensor field A on \mathcal{M} with the property that, in any local coordinate system (x^1, \dots, x^n) on \mathcal{M}

$$A_{ij}^k = \Gamma_{ij}^k.$$

Hint: Check how Γ_{ij}^k transforms under changes of coordinates.

- (b) Show that the torsion $T : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{M})$ of the connection ∇ , which is defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

is a tensor field.

- (c) Let $\bar{\nabla}$ be a (possibly) different connection on \mathcal{M} . Show that the difference $\nabla - \bar{\nabla} : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{M})$ is also a tensor field. Deduce that, there exists a (1, 2)-type tensor field A such that, in any given local coordinate system (x^1, \dots, x^n) ,

$$A_{ij}^k = \Gamma_{ij}^k - \bar{\Gamma}_{ij}^k$$

where Γ_{ij}^k and $\bar{\Gamma}_{ij}^k$ are the Christoffel symbols of ∇ and $\bar{\nabla}$, respectively.

Solution. (a) Assume that there exists a tensor field A as in the statement. Then, if (x^1, \dots, x^n) and (y^1, \dots, y^n) are two coordinate systems around the same point $p \in \mathcal{M}$, the components A_{ij}^k and \tilde{A}_{ij}^k of A in the two coordinate systems, respectively, are related by the transformation formula

$$\tilde{A}_{ij}^k = A_{\alpha\beta}^\gamma \cdot \frac{\partial y^k}{\partial x^\alpha} \frac{\partial x^\beta}{\partial y^i} \frac{\partial x^\beta}{\partial y^j}. \quad (1)$$

On the other hand, the Christoffel symbols Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ in the coordinate systems (x^1, \dots, x^n) and (y^1, \dots, y^n) , respectively, are given by the relations

$$\Gamma_{ij}^k = dx^k \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)$$

and

$$\begin{aligned} \tilde{\Gamma}_{ij}^k &= dy^k \left(\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} \right) \\ &= \frac{\partial y^k}{\partial x^\gamma} dx^\gamma \left(\nabla_{\frac{\partial x^\alpha}{\partial y^i} \cdot \frac{\partial}{\partial x^\alpha}} \left(\frac{\partial x^\beta}{\partial y^j} \cdot \frac{\partial}{\partial x^\beta} \right) \right) \\ &= \frac{\partial y^k}{\partial x^\gamma} \cdot \frac{\partial x^\alpha}{\partial y^i} \cdot dx^\gamma \left(\nabla_{\frac{\partial}{\partial x^\alpha}} \left(\frac{\partial x^\beta}{\partial y^j} \cdot \frac{\partial}{\partial x^\beta} \right) \right) \\ &= \frac{\partial y^k}{\partial x^\gamma} \cdot \frac{\partial x^\alpha}{\partial y^i} \cdot dx^\gamma \left(\frac{\partial}{\partial x^\alpha} \left(\frac{\partial x^\beta}{\partial y^j} \right) \cdot \frac{\partial}{\partial x^\beta} + \frac{\partial x^\beta}{\partial y^j} \cdot \nabla_{\frac{\partial}{\partial x^\alpha}} \left(\frac{\partial}{\partial x^\beta} \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{\partial y^k}{\partial x^\gamma} \cdot \frac{\partial x^\alpha}{\partial y^i} \cdot \left(\frac{\partial}{\partial x^\alpha} \left(\frac{\partial x^\beta}{\partial y^j} \right) \cdot dx^\gamma \left(\frac{\partial}{\partial x^\beta} \right) + \frac{\partial x^\beta}{\partial y^j} \cdot dx^\gamma \left(\nabla_{\frac{\partial}{\partial x^\alpha}} \left(\frac{\partial}{\partial x^\beta} \right) \right) \right) \\
 &= \frac{\partial y^k}{\partial x^\gamma} \cdot \frac{\partial x^\alpha}{\partial y^i} \cdot \frac{\partial}{\partial x^\alpha} \left(\frac{\partial x^\gamma}{\partial y^j} \right) + \Gamma_{\alpha\beta}^\gamma \cdot \frac{\partial y^k}{\partial x^\gamma} \cdot \frac{\partial x^\alpha}{\partial y^i} \cdot \frac{\partial x^\beta}{\partial y^j}
 \end{aligned}$$

(note that we used the fact that that $dx^k(\cdot)$ is a tensor field and, thus, is $C^\infty(\mathcal{M})$ -linear in its argument). Therefore, we see that the transformation law for the Christoffel symbols contains an additional term which is not there in (1), namely $\frac{\partial y^k}{\partial x^\gamma} \cdot \frac{\partial x^\alpha}{\partial y^i} \cdot \frac{\partial}{\partial x^\alpha}$. Expressing the coordinates $y^i = y^i(x)$ as functions of (x^1, \dots, x^n) , this term is equal to

$$[Dy]_\gamma^k \cdot ([Dy]^{-1})_i^\alpha \cdot \left(\frac{\partial}{\partial x^\alpha} ([Dy]^{-1})_j^\beta \right)$$

where $[DY]_\alpha^i = \frac{\partial y^i}{\partial x^\alpha}$ is the Jacobian matrix for y . In particular, if the second derivatives of the transformation $x \rightarrow y(x)$ at $p \in \mathcal{M}$ are *not* all 0, then this term will have a non-zero at p . Therefore, Γ_{ij}^k does not transform under coordinate changes like a tensor field.

(b) In order to show that T is a tensor field, it suffices to show that it is $C^\infty(\mathcal{M})$ -linear in its arguments; since T obviously satisfies $T(X_1 + X_2, Y) = T(X_1, Y) + T(X_2, Y)$ (because ∇ and $[\cdot, \cdot]$ are \mathbb{R} -linear in their arguments) and $T(X, Y) = -T(Y, X)$, it suffices to show that, for any $X, Y \in \Gamma(\mathcal{M})$ and $f \in C^\infty(\mathcal{M})$:

$$T(fX, Y) = fT(X, Y).$$

Recall that the Lie bracket $[\cdot, \cdot]$ satisfies for any

$$[fX, Y] = f[X, Y] - Y(f) \cdot X$$

since, for any $h \in C^\infty(\mathcal{M})$:

$$[fX, Y](h) = fX(Y(h)) - Y(fX(h)) = fX(Y(h)) - Y(f)X(h) - fY(X(h)) = f[X, Y](h) - Y(f)X(h).$$

Using the above observation and the fact that ∇ is $C^\infty(\mathcal{M})$ in its first argument and satisfies the Leibniz rule with respect to its second argument, we can calculate:

$$\begin{aligned}
 T(fX, Y) &= \nabla_{fX}Y - \nabla_Y(fX) - [fX, Y] \\
 &= f\nabla_XY - Y(f)X - f\nabla_YX - f[X, Y] + Y(f)X \\
 &= f \cdot (\nabla_{fX}Y - \nabla_Y(fX) - [X, Y]) \\
 &= fT(X, Y).
 \end{aligned}$$

(c) As before, we have to verify that $\nabla - \bar{\nabla}$ is $C^\infty(\mathcal{M})$ -linear in both its arguments; since, by the definition of a connection, both ∇ and $\bar{\nabla}$ are $C^\infty(\mathcal{M})$ -linear in their first argument and \mathbb{R} -linear in their second argument, it remains to prove that, for any $X, Y \in \Gamma(\mathcal{M})$ and $f \in C^\infty(\mathcal{M})$:

$$(\nabla - \bar{\nabla})(X, fY) = f(\nabla - \bar{\nabla})(X, Y).$$

Indeed:

$$\begin{aligned}
 (\nabla - \bar{\nabla})(X, fY) &= \nabla_X(fY) - \bar{\nabla}_X(fY) \\
 &= X(f)Y + f\nabla_X Y - X(f)Y - f\bar{\nabla}_X Y \\
 &= f\nabla_X Y - f\bar{\nabla}_X Y \\
 &= f(\nabla - \bar{\nabla})(X, Y).
 \end{aligned}$$

Therefore, setting $A(X, Y) \doteq (\nabla - \bar{\nabla})(X, Y) = \nabla_X Y - \bar{\nabla}_X Y$, we have shown that $A : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{M})$ is a $(1, 2)$ -tensor field; it is easy to verify that, in any local coordinate system (x^1, \dots, x^n) , the components A_{ij}^k of A take the form

$$A_{ij}^k = \Gamma_{ij}^k - \bar{\Gamma}_{ij}^k.$$

5.2 Let $\Psi : \mathcal{M}^n \rightarrow \mathbb{R}^{n+1}$ be an immersion such that $\Psi(\mathcal{M})$ is a *spacelike* hypersurface of (\mathbb{R}^{n+1}, η) and let $\bar{g} = \Psi_*\eta$ be the induced metric. Let (x^1, \dots, x^n) be a local coordinate chart on \mathcal{M} . Compute the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection associated to \bar{g} in the (x^1, \dots, x^n) coordinates as functions of Ψ and its derivatives.

Solution. Let (x^1, \dots, x^n) be a local coordinate chart on \mathcal{M} . The components \bar{g}_{ij} of the induced metric \bar{g} on this chart takes the form

$$\bar{g}_{ij} = \bar{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \eta\left(\Psi^*\left(\frac{\partial}{\partial x^i}\right), \Psi^*\left(\frac{\partial}{\partial x^j}\right)\right) = \eta_{\alpha\beta} \frac{\partial \Psi^\alpha}{\partial x^i} \frac{\partial \Psi^\beta}{\partial x^j}.$$

We can then compute

$$\frac{\partial \bar{g}_{jk}}{\partial x^i} = \eta_{\alpha\beta} \frac{\partial^2 \Psi^\alpha}{\partial x^i \partial x^j} \frac{\partial \Psi^\beta}{\partial x^k} + \eta_{\alpha\beta} \frac{\partial \Psi^\alpha}{\partial x^j} \frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^k}.$$

Since we assumed that $\Psi(\mathcal{M})$ is spacelike, the induced metric \bar{g} is *Riemannian*. In particular, the matrix $[\bar{g}_{ij}]$ is invertible; denoting with \bar{g}^{ij} the components of the inverse matrix of \bar{g}_{ij} , we can readily compute

$$\begin{aligned}
 \Gamma_{ij}^k &= \frac{1}{2} \bar{g}^{kl} (\partial_i \bar{g}_{lj} + \partial_j \bar{g}_{li} - \partial_l \bar{g}_{ij}) \\
 &= \frac{1}{2} \bar{g}^{kl} \eta_{\alpha\beta} \left(\frac{\partial^2 \Psi^\alpha}{\partial x^i \partial x^l} \frac{\partial \Psi^\beta}{\partial x^j} + \frac{\partial \Psi^\alpha}{\partial x^l} \frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^j} + \frac{\partial^2 \Psi^\alpha}{\partial x^j \partial x^l} \frac{\partial \Psi^\beta}{\partial x^i} + \frac{\partial \Psi^\alpha}{\partial x^l} \frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^j} \right. \\
 &\quad \left. - \frac{\partial^2 \Psi^\alpha}{\partial x^i \partial x^l} \frac{\partial \Psi^\beta}{\partial x^j} - \frac{\partial \Psi^\alpha}{\partial x^i} \frac{\partial^2 \Psi^\beta}{\partial x^j \partial x^l} \right) \\
 &= \bar{g}^{kl} \eta_{\alpha\beta} \frac{\partial \Psi^\alpha}{\partial x^l} \frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^j}
 \end{aligned}$$

(where, in passing to the last line above, we used the fact that $\eta_{\alpha\beta}$ is symmetric in α, β to write $\eta_{\alpha\beta} \frac{\partial^2 \Psi^\alpha}{\partial x^j \partial x^l} \frac{\partial \Psi^\beta}{\partial x^i} = \eta_{\alpha\beta} \frac{\partial \Psi^\alpha}{\partial x^i} \frac{\partial^2 \Psi^\beta}{\partial x^j \partial x^l}$).

Remark. An alternative way to view the above formula is as follows: Since $\Psi(\mathcal{M})$ is a spacelike hypersurface of (\mathbb{R}^{n+1}, η) of dimension n , for any $p \in \mathcal{M}$ the tangent space $T_{\Psi(p)}\Psi(\mathcal{M})$ (which is simply the image of $d\Psi : T_p\mathcal{M} \rightarrow T_{\Psi(p)}\mathbb{R}^{n+1}$) is a spacelike hyperplane of $T_{\Psi(p)}\mathbb{R}^{n+1}$ of codimension 1. Let us denote with $\Pi_p^\top : T_{\Psi(p)}\mathbb{R}^{n+1} \rightarrow T_p\mathcal{M}$ the composition of the orthogonal projection (with respect to η) $T_{\Psi(p)}\mathbb{R}^{n+1} \rightarrow T_{\Psi(p)}\Psi(\mathcal{M})$ with the linear isomorphism $(d\Psi)^{-1} : T_{\Psi(p)}\Psi(\mathcal{M}) \rightarrow T_p\mathcal{M}$. Then, it is easy to verify that the map Π_p^\top expressed with respect the Cartesian frame $\{e_\alpha\}_{\alpha=0}^n$ on $T_{\Psi(p)}\mathbb{R}^{n+1}$ and the $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ frame on $T_p\mathcal{M}$ takes the form

$$(\Pi_p^\top)_\alpha^k = \bar{g}^{kl}|_p \cdot \eta_{\alpha\beta} \cdot \frac{\partial \Psi^\beta}{\partial x^l}(p)$$

(you should be able to verify this by noting that Π_p^\top maps $\Psi_*(\partial_i) = \frac{\partial \Psi^\alpha}{\partial x^i} e_\alpha$ to ∂_i and any vector η -orthogonal to $T_{\Psi(p)}\Psi(\mathcal{M}) = \text{span}\{\Psi^*(\partial_i)\}_{i=1}^n$ is mapped to 0). Then, the above expression for the Christoffel symbols of \bar{g} can be reexpressed as

$$\Gamma_{ij}^k = (\Pi^\top)_\beta^k \frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^j}.$$

Note that, since $\Psi_*(\partial_i) = \frac{\partial \Psi^\alpha}{\partial x^i} e_\alpha$, the term $\frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^j}$ is simply the α -th component of $\nabla_{\Psi_*(\partial_i)}^{(\eta)} \Psi_*(\partial_j)$, where $\nabla^{(\eta)}$ is the flat connection on \mathbb{R}^{n+1} . Thus, the induced connection on \mathcal{M} via Ψ is just the *orthogonal projection* onto $\Psi(\mathcal{M})$ of the flat connection on \mathbb{R}^{n+1} .

5.3 Let \mathcal{M} be a smooth manifold equipped with a connection ∇ . We can extend the connection ∇ to a map $\nabla : \Gamma(M) \times \text{Ten}_l^k(\mathcal{M}) \rightarrow \text{Ten}_l^k(\mathcal{M})$ by the requirements that

- ∇ satisfies the Leibniz rule with respect to tensor products, i.e. for all $X \in \Gamma(M)$

$$\nabla_X(f \otimes g) = \nabla_X f \otimes g + f \otimes \nabla_X g,$$

- ∇ commutes with contractions, i.e.

$$\nabla_X(\text{tr}A) = \text{tr}(\nabla_X A).$$

Show that, in any local coordinate chart (x^1, \dots, x^n) , if Γ_{ij}^k are the Christoffel symbols of ∇ then, for every 1-form ω :

$$(\nabla_{\frac{\partial}{\partial x^i}} \omega)_j = \partial_i \omega_j - \Gamma_{ij}^k \omega_k.$$

Moreover, for any (k, l) -tensor field T :

$$\begin{aligned} (\nabla_{\frac{\partial}{\partial x^a}} T)^{i_1 \dots i_k}_{j_1 \dots j_l} &= \partial_a T^{i_1 \dots i_k}_{j_1 \dots j_l} + \Gamma_{ab}^{i_1} T^{bi_2 \dots i_k}_{j_1 \dots j_l} + \dots + \Gamma_{ab}^{i_k} T^{i_1 \dots i_{k-1} b}_{j_1 \dots j_l} \\ &\quad - \Gamma_{aj_1}^b T^{bi_2 \dots i_k}_{bj_2 \dots j_l} - \dots - \Gamma_{aj_l}^b T^{i_1 \dots i_{k-1} b}_{j_1 \dots j_{l-1} b}. \end{aligned}$$

Solution. We will start by observing that, for any 1-form ω and any vector field X on \mathcal{M} , the function $\omega(X) \in C^\infty(\mathcal{M})$ can be seen as the contraction $\text{tr}(\omega \otimes X)$ of the $(1, 1)$ -tensor field $\omega \otimes X$; this can be seen clearly in local coordinates, since

$$(\omega \otimes X)_j^i \doteq \omega_i X^j \quad \text{and} \quad \omega(X) = \omega_k X^k.$$

Therefore, using our assumptions that $\nabla_X(f \otimes h) = \nabla_X f \otimes h + f \otimes \nabla_X h$ and ∇ commutes with contractions, we obtain for any $X, Y \in \Gamma(\mathcal{M})$:

$$\begin{aligned} Y(\omega(X)) &= Y(\text{tr}(\omega \otimes X)) \\ &= \text{tr}(\nabla_Y(\omega \otimes X)) \\ &= \text{tr}(\nabla_Y \omega \otimes X + \omega \otimes \nabla_Y X) \\ &= \nabla_Y \omega(X) + \omega(\nabla_Y X). \end{aligned}$$

By rearranging the terms in the above identity, we thus obtain:

$$\nabla_Y \omega(X) = Y(\omega(X)) - \omega(\nabla_Y X).$$

In any given local coordinate system (x^1, \dots, x^n) on \mathcal{M} , if we apply the above formula for $X = \frac{\partial}{\partial x^j}$ and $Y = \frac{\partial}{\partial x^i}$ we obtain:

$$\begin{aligned} \left(\nabla_{\frac{\partial}{\partial x^i}} \omega\right)_j &= \partial_i(\omega_j) - (\nabla_{\partial_i} \partial_j)^k \omega_k \\ &= \partial_i(\omega_j) - \Gamma_{ij}^k \omega_k. \end{aligned}$$

In particular, if $\omega = dx^k$ is a coordinate 1-form, then

$$\nabla_{\partial_i}(dx^k) = -\Gamma_{ij}^k dx^j.$$

If T is a tensor field of type (k, l) , then it can be expressed in a local coordinate system (x^1, \dots, x^n) as before as a linear combination of the coordinate (k, l) -tensor fields $\frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^{\delta_1} \otimes \dots \otimes dx^{\delta_l}$, $\gamma_1, \dots, \gamma_k, \delta_1, \dots, \delta_l \in \{1, \dots, n\}$:

$$T = T^{\gamma_1 \dots \gamma_k}_{\delta_1 \dots \delta_l} \frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^{\delta_1} \otimes \dots \otimes dx^{\delta_l}. \quad (2)$$

Our assumption on the behaviour of ∇ on tensor products and the fact that ∇ satisfies the Leibniz rule implies that, for any $f \in C^\infty(\mathcal{M})$, any $X \in \Gamma(\mathcal{M})$ and any $(Y_{(1)}, \dots, Y_{(k)}, \omega_{(1)}, \dots, \omega_{(l)}) \in \Gamma(\mathcal{M}) \times \dots \times \Gamma(\mathcal{M}) \times \Gamma^*(\mathcal{M}) \times \dots \times \Gamma^*(\mathcal{M})$, we have

$$\begin{aligned} \nabla_X(f Y_{(1)} \otimes \dots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \dots \otimes \omega_{(l)}) &= X(f) Y_{(1)} \otimes \dots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \dots \otimes \omega_{(l)} \\ &\quad + f (\nabla_X Y_{(1)}) \otimes \dots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \dots \otimes \omega_{(l)} \\ &\quad + \dots + f Y_{(1)} \otimes \dots \otimes (\nabla_X Y_{(k)}) \otimes \omega_{(1)} \otimes \dots \otimes \omega_{(l)} \\ &\quad + f Y_{(1)} \otimes \dots \otimes Y_{(k)} \otimes \nabla_X(\omega_{(1)}) \otimes \dots \otimes \omega_{(l)} \\ &\quad + \dots + f Y_{(1)} \otimes \dots \otimes Y_{(k)} \otimes \omega_{(1)} \otimes \dots \otimes (\nabla_X \omega_{(l)}). \end{aligned}$$

Therefore, applying this formula for the $\nabla_{\frac{\partial}{\partial x^\alpha}}$ derivative of the expression (2) and using the fact that

$$\nabla_{\partial_\alpha} \frac{\partial}{\partial x^i} = \Gamma_{\alpha i}^j \frac{\partial}{\partial x^j}, \quad \nabla_{\partial_\alpha} (dx^i) = -\Gamma_{\alpha j}^i dx^j$$

(the last formula following from our computation of the expression of ∇ acting on 1-forms), we obtain:

$$\begin{aligned} \nabla_{\partial_\alpha} T &= (\partial_\alpha T^{\gamma_1 \dots \gamma_k}_{\delta_1 \dots \delta_l}) \frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^{\delta_1} \otimes \dots \otimes dx^{\delta_l} \\ &+ T^{\gamma_1 \dots \gamma_k}_{\delta_1 \dots \delta_l} \Gamma_{\alpha \gamma_1}^\beta \frac{\partial}{\partial x^\beta} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^{\delta_1} \otimes \dots \otimes dx^{\delta_l} \\ &+ \dots + T^{\gamma_1 \dots \gamma_k}_{\delta_1 \dots \delta_l} \Gamma_{\alpha \gamma_k}^\beta \frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^\beta} \otimes dx^{\delta_1} \otimes \dots \otimes dx^{\delta_l} \\ &- T^{\gamma_1 \dots \gamma_k}_{\delta_1 \dots \delta_l} \Gamma_{\alpha \beta}^{\delta_1} \frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^\beta \otimes \dots \otimes dx^{\delta_l} \\ &- \dots - T^{\gamma_1 \dots \gamma_k}_{\delta_1 \dots \delta_l} \Gamma_{\alpha \beta}^{\delta_l} \frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^{\delta_1} \otimes \dots \otimes dx^\beta. \end{aligned}$$

Therefore, considering the $\frac{\partial}{\partial x^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\gamma_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l}$ component of the above expression (noticing that, in each summand involving Γ , an index of Γ is contracted with one index of T , and we are free to rename those indices as we please), we obtain

$$\begin{aligned} (\nabla_{\frac{\partial}{\partial x^\alpha}} T)^{i_1 \dots i_k}_{j_1 \dots j_l} &= \partial_\alpha T^{i_1 \dots i_k}_{j_1 \dots j_l} + \Gamma_{ab}^{i_1} T^{bi_2 \dots i_k}_{j_1 \dots j_l} + \dots + \Gamma_{ab}^{i_k} T^{i_1 \dots i_{k-1} b}_{j_1 \dots j_l} \\ &- \Gamma_{aj_1}^b T^{i_1 i_2 \dots i_k}_{bj_2 \dots j_l} - \dots - \Gamma_{aj_l}^b T^{i_1 \dots i_k}_{j_1 \dots j_{l-1} b}. \end{aligned}$$

5.4 Let $(\overline{\mathcal{M}}, \bar{g})$ be a *Riemannian* manifold (i.e. \bar{g} is positive definite) and let us define the Lorentzian manifold (\mathcal{M}, g) so that $\mathcal{M} = \mathbb{R} \times \overline{\mathcal{M}}$ and g is the product metric $g = -(dt)^2 + \bar{g}$; this means that, for every local coordinate chart (x^1, \dots, x^n) on $\mathcal{U} \subset \overline{\mathcal{M}}$, if we extend it to a local coordinate chart (t, x^1, \dots, x^n) on $\mathbb{R} \times \mathcal{U} \subset \mathcal{M}$ so that t is simply the projection on the \mathbb{R} factor, then

$$g = -dt^2 + \bar{g}_{ij} dx^i dx^j.$$

Show that a curve $\gamma : (0, 1) \rightarrow (\mathcal{M}, g)$ is a geodesic (for the Levi-Civita connection of g) if and only, in any local coordinate system $(t; x^1, \dots, x^n)$ as above, if it can be written in the form

$$\gamma(s) = (t(s); \bar{\gamma}^i(s))$$

where $t(s) = \lambda_1 s + \lambda_0$ for some $\lambda_1, \lambda_0 \in \mathbb{R}$ and $\bar{\gamma} : (0, 1) \rightarrow \overline{\mathcal{M}}$ is a geodesic of $(\overline{\mathcal{M}}, \bar{g})$.

Solution. For any $p \in \mathcal{M}$, let $(x^0, x^1, \dots, x^n) = (t, x^1, \dots, x^n)$ be a local coordinate system around p which is as described in the statement of the exercise. We will adopt the following convention: We will use Greek letters (i.e. $\alpha, \beta, \gamma, \dots$) for indices ranging from 0 to n and Latin letters (i.e. i, j, k, \dots) for indices ranging from 1 to n . With this notation, the components $g_{\alpha\beta}$ of g take the form

$$g_{00} = -1, \quad g_{0i} = 0, \quad g_{ij} = \bar{g}_{ij}.$$

Therefore, we can also calculate that the components $g^{\alpha\beta}$ of the inverse matrix of $[g]$ take the form

$$g^{00} = -1, \quad g^{0i} = 0, \quad g^{ij} = \bar{g}^{ij}$$

(where \bar{g}^{ij} are the components of the inverse matrix of $[\bar{g}_{ij}]$).

Let us, now turn to calculating the Christoffel symbols $\Gamma_{\alpha\beta}^{\delta}$ of g . Using the formula

$$\Gamma_{\alpha\beta}^{\delta} = \frac{1}{2}g^{\delta\lambda}(\partial_{\alpha}g_{\lambda\beta} + \partial_{\beta}g_{\lambda\alpha} - \partial_{\lambda}g_{\alpha\beta}),$$

we can readily verify that

$$\Gamma_{\alpha\beta}^{\delta} = 0 \quad \text{when at least one of } \alpha, \beta, \delta \text{ is } 0$$

and

$$\Gamma_{ij}^k = \bar{\Gamma}_{ij}^k \quad \text{when } i, j, k \in \{1, \dots, n\},$$

where $\bar{\Gamma}_{ij}^k$ are the Christoffel symbols of \bar{g} .

Let $\gamma : (0, 1) \rightarrow \mathcal{M}$, $\gamma(s) = (x^0(s), x^1(s), \dots, x^n(s))$ be a geodesic of g . Thus, the components of γ satisfy the geodesic ODE

$$\ddot{x}^{\alpha} + \Gamma_{\beta\delta}^{\alpha}\dot{x}^{\beta}\dot{x}^{\delta} = 0.$$

Applying the above relation for $\alpha = 0$ and using the fact that (as we calculated) $\Gamma_{\beta\gamma}^0 = 0$, we obtain

$$\ddot{x}^0(s) = 0 \Rightarrow x^0(s) = \lambda_1 s + \lambda_0 \quad \text{for some } \lambda_0, \lambda_1 \in \mathbb{R}.$$

Similarly, applying the above relation for $\alpha = k \in \{1, \dots, n\}$, we obtain

$$\begin{aligned} 0 &= \ddot{x}^k + \Gamma_{\beta\delta}^k \dot{x}^{\beta} \dot{x}^{\delta} \\ &= \ddot{x}^k + \Gamma_{ij}^k \dot{x}^i \dot{x}^j + \Gamma_{i0}^k \dot{x}^i \dot{x}^0 + \Gamma_{0j}^k \dot{x}^0 \dot{x}^j + \Gamma_{00}^k \dot{x}^0 \dot{x}^0 \\ &= \ddot{x}^k + \bar{\Gamma}_{ij}^k \dot{x}^i \dot{x}^j + 0, \end{aligned}$$

i.e. the curve $s \rightarrow (x^1(s), \dots, x^n(s))$ satisfies the geodesic equation with respect to the metric \bar{g} .

Remark. The above proof can be easily generalised to the case of a pseudo-Riemannian manifold (\mathcal{M}, g) which is the product of the pseudo-Riemannian manifolds (\mathcal{M}_1, g_1) and (\mathcal{M}_2, g_2) . In that case, the projections γ_1 and γ_2 of any geodesic γ of (\mathcal{M}, g) on \mathcal{M}_1 and \mathcal{M}_2 , respectively, are geodesics for g_1 and g_2 ; the proof uses the fact that, in any product coordinate system $(x^1, \dots, x^{n_1}; y^1, \dots, y^{n_2})$ on \mathcal{M} (where (x^1, \dots, x^{n_1}) and (y^1, \dots, y^{n_2}) are local coordinates on \mathcal{M}_1 and \mathcal{M}_2 , respectively), any Christoffel symbol $\Gamma_{\alpha\beta}^{\delta}$ with *mixed* indices (i.e. with indices belonging to both (x^1, \dots, x^{n_1}) and (y^1, \dots, y^{n_2})) has to vanish.

5.5 In this exercise, we will prove that there exist compact Lorentzian manifolds which are **geodesically incomplete** (recall that, as a consequence of the Hopf–Rinow theorem in Riemannian geometry, every compact Riemannian manifold is geodesically complete). Consider the manifold $\mathcal{M} = \mathbb{R}^2 \setminus 0$ equipped with the metric

$$g = \frac{1}{u^2 + v^2} du dv.$$

- (a) Verify that (\mathcal{M}, g) is a smooth Lorentzian manifold and that the map $(u, v) \rightarrow (\lambda \cdot u, \lambda \cdot v)$ is an isometry for every $\lambda \neq 0$.
- (b) Consider the group of isometries $\Gamma = \{(u, v) \rightarrow (2^k u, 2^k v), k \in \mathbb{Z}\}$. Show that the quotient space \mathcal{M}/Γ is a compact manifold. Show also that \mathcal{M}/Γ inherits a natural metric \tilde{g} from (\mathcal{M}, g) so that the quotient map $(\mathcal{M}, g) \rightarrow (\mathcal{M}/\Gamma, \tilde{g})$ is a local isometry.
- (c) Show that the map $(\mathcal{M}, g) \rightarrow (\mathcal{M}/\Gamma, \tilde{g})$ maps geodesics to geodesics. Compute the geodesic equation on (\mathcal{M}, g) and deduce that $(\mathcal{M}/\Gamma, \tilde{g})$ contains a geodesic $\gamma : (a, b) \rightarrow \mathcal{M}/\Gamma$ with $b < +\infty$ which *cannot* be extended beyond $t = b$.

Solution. (a) It is straightforward to verify that g is a smooth Lorentzian metric on the smooth manifold $\mathcal{M} = \mathbb{R}^2 \setminus 0$ (in fact, it is conformal to the Minkowski metric on $\mathbb{R}^2 \setminus 0$). For any $\lambda \in \mathbb{R} \setminus 0$, we can readily compute that the map $T_\lambda : \mathcal{M} \rightarrow \mathcal{M}$, defined by

$$T_\lambda(u, v) = (\lambda u, \lambda v),$$

is a diffeomorphism satisfying

$$\begin{aligned} (T_\lambda)_*g &= \frac{1}{(\lambda u)^2 + (\lambda v)^2} d(\lambda u)d(\lambda v) \\ &= \frac{1}{u^2 + v^2} dudv \\ &= g. \end{aligned}$$

Therefore, T_λ is an isometry of (\mathcal{M}, g)

(b) Let's recall first a few things about the quotient of a manifold by a subgroup of diffeomorphisms: Let G be a subgroup of $\text{Diff}(\mathcal{N})$ for a smooth manifold \mathcal{N} . Setting, for any point $p \in \mathcal{N}$,

$$[p]_G \doteq \{q \in \mathcal{N} : q = F(p) \text{ for some } F \in G\},$$

then the set

$$\mathcal{N}/G \doteq \{[p]_G : p \in \mathcal{N}\}$$

(which is called *the quotient* of \mathcal{N} by the action of G), equipped with the quotient topology, has the structure of a smooth manifold if and only if, for any $p \in \mathcal{N}$, there exists an open neighborhood $\mathcal{U} \subset \mathcal{N}$ of p such that

$$\mathcal{U} \cap F(\mathcal{U}) = \emptyset \quad \text{for all } F \in G \tag{3}$$

(it is straightforward to verify that, for any $p \in \mathcal{M}$, if $\Phi : \mathcal{V} \rightarrow \mathbb{R}^n$ is a smooth coordinate chart on a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p , then the collection of coordinate charts

$$\tilde{\Phi} = \{\Phi \circ F^{-1} : F \in G\}$$

is a G -invariant set of coordinate charts on neighborhoods of all the points in $[p]_G$ and can be used to construct a coordinate chart around $[p]_G$ in \mathcal{N}/G). With this manifold structure on \mathcal{N}/G , the

Differential Geometry IV: General Relativity

EPFL– Fall 2025
SOLUTIONS: Series 5

G. Moschidis
10 Oct. 2025

quotient map $\pi : \mathcal{N} \rightarrow \mathcal{N}/G$, $p \rightarrow [p]_G$, is a local (but not global) diffeomorphism. Notice that, for any curve $\gamma : (a, b) \rightarrow \mathcal{N}/G$, the preimage of γ in \mathcal{N} consists of the family of curves

$$\pi^{-1}(\gamma) = \bigcup_{F \in G} \gamma_F,$$

satisfying

$$F_1(\gamma_{F_2}) = \gamma_{F_1 \circ F_2}, \quad \text{for all } F_1, F_2 \in G.$$

By considering the tangent vectors to such curves, we infer the following statement about $T(\mathcal{N}/G)$: For any $p \in \mathcal{N}$ and any tangent vector $v \in T_{[p]_G}(\mathcal{N}/G)$, there exists a family of tangent vectors $v_F \in T_{F(p)}\mathcal{N}$, $F \in G$, such that

$$\pi^*(v_F) = v \quad \text{for all } F \in G$$

and satisfying

$$F_1^*(v_{F_2}) = v_{F_1 \circ F_2} \quad \text{for any } F_1, F_2 \in G.$$

Returning to our case (where $\mathcal{N} = \mathcal{M}$ and $G = \Gamma$), in order to verify that \mathcal{M}/Γ is a *compact* manifold, it suffices to show that there exists a compact subset $\mathcal{K} \subset \mathcal{M}$ such that the quotient map π is onto when restricted to \mathcal{K} (compactness of \mathcal{M}/Γ in this case follows from the fact that, since π is continuous, $\pi(\mathcal{K})$ is necessarily compact). We can readily verify that

$$\mathcal{K} = \{(u, v) \in \mathbb{R}^2 \setminus 0 : \frac{1}{2} \leq u^2 + v^2 \leq 2\}$$

has this property (which can be equivalently reexpressed as the statement that, for every $p \in \mathbb{R}^2 \setminus 0$, there exists an $F \in \Gamma$ such that $F(p) \in \mathcal{K}$).

We will now use the fact that Γ is in fact a group of isometries to deduce that the quotient manifold \mathcal{M}/Γ admits a *quotient metric* \tilde{g} . It is natural to define, for any $[p]_G \in \mathcal{M}/\Gamma$ and any $v, w \in T_{[p]_G}\mathcal{M}/\Gamma$,

$$\tilde{g}(v, w) \doteq g(v_F, w_F) \quad \text{for all } F \in \Gamma \tag{4}$$

(see the the discussion above for the notation v_F, w_F). The above definition, of course, makes sense only when the right hand side of (4) is the same for all $F \in \Gamma$; this is true precisely when Γ is a group of isometries of (\mathcal{M}, g) , since then $g(v_F, w_F) = g(F^*v_1, F^*v_2)$ is equal to $g(v_1, w_1)$ (1 being the identity element in Γ). Moreover, since $\pi^*(v_F) = v$, (4) trivially implies that, in this case, the quotient map π is a local isometry.

Remark. The above argument works in the case of any group of isometries G acting on a pseudoriemannian manifold (\mathcal{N}, g) in a way that (3) holds.

(c) In general, if $\Psi : (\mathcal{N}_1, g_1) \rightarrow (\mathcal{N}_2, g_2)$ is a local isometry, then, for any $X, Y \in \Gamma(\mathcal{N}_1)$, we have $\nabla_{\Psi^*X}^{(\mathcal{N}_2)}(\Psi^*Y) = \nabla_X^{(\mathcal{N}_1)}Y$ (where $\nabla^{(\mathcal{N}_i)}$ denotes the Levi-Civita connection of (\mathcal{N}_i, g_i)); this can be readily verified using the formula of Kozul for any vector fields U, V, W :

$$2g_i(\nabla_U^{(\mathcal{N}_i)}V, W) = U(g(V, W)) + V(g(U, W)) - W(g(U, V)) - g([V, W], U) - g([U, W], V) + g([U, V], W)$$

(using $U = X, V = Y, W = Z \in \Gamma(\mathcal{N}_1)$ for $i = 1$ and $U = \Psi^*X, V = \Psi^*Y, W = \Psi^*Z$ for $i = 2$). Thus, if $\gamma : I \rightarrow \mathcal{N}_1$ is a geodesic of g_1 , i.e. satisfies $\nabla_{\dot{\gamma}}^{(\mathcal{N}_1)}\dot{\gamma} = 0$, then $\nabla_{\Psi^*\dot{\gamma}}^{(\mathcal{N}_2)}(\Psi^*\dot{\gamma}) = 0$, i.e. $\Psi(\dot{\gamma})$

is a geodesic of (\mathcal{N}_2, g_2) . Therefore, since, in our case, the quotient map $\pi : \mathcal{M} \rightarrow \mathcal{M}/\Gamma$ is a local isometry, it maps geodesics to geodesics.

In the (u, v) coordinate system on \mathcal{M} , we can readily compute that the Christoffel symbols of the Levi-Civita connection of g take the following form:

$$\Gamma_{uu}^u = -\frac{2u}{u^2 + v^2}, \quad \Gamma_{vv}^v = -\frac{2v}{u^2 + v^2}, \quad \Gamma_{uv}^u = \Gamma_{vv}^u = \Gamma_{uv}^v = \Gamma_{uu}^v = 0.$$

Therefore, the geodesic equation takes the following form: If $s \rightarrow (u(s), v(s))$ is a geodesic of (\mathcal{M}, g) , then

$$\begin{aligned} \ddot{u} - \frac{2u}{u^2 + v^2}(\dot{u})^2 &= 0, \\ \ddot{v} - \frac{2v}{u^2 + v^2}(\dot{v})^2 &= 0. \end{aligned}$$

It can be easily verified that the curve $s \rightarrow (u(s), v(s)) = (\frac{1}{s}, 0)$, $s \in (-\infty, 0)$ is a null geodesic of (\mathcal{M}, g) , which is maximally extended (since, as a subset of \mathbb{R}^2 , the limit point of this curve is $(0, 0)$ as $s \rightarrow 0$). The projection of this curve on \mathcal{M}/Γ is, therefore, a maximally extended geodesic of \tilde{g} .